CHAPTER 4
MULTIDIMENSIONAL UNREPLICATED LINEAR FUNCTIONAL RELATIONSHIP MODEL WITH SINGLE SLOPE AND ITS COEFFICIENT OF DETERMINATION

Issues or problems stated in previous chapters will be investigated in Chapter 4 and Chapter 5. For example, the COD for the ULFR model was shown to be suitable for use on imperfect reference image only. Henceforth, the performance of the COD will be investigated when we have multiple image attributes and when local and global information are combined. The multivariate version of the ULFR model will then be investigated.

This study considers a new functional relationship model where the $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ are p-dimensional vectors, and are linearly related. It can be considered as a generalization of the unreplicated linear functional relationship model proposed by Adcock (1877) where the bivariate observations are only one-dimensional. The essential differences of the proposed model and the multivariate model of Chan & Mak (1983, 1984) involved the different elements in the intercept vector $\boldsymbol{\alpha} = \left(\alpha_1, \alpha_2, \ldots, \alpha_p\right)'$ and the slope matrix $\boldsymbol{B}$ is replaced by a scalar value β. Practically, a quality attribute measured from reference image can only be compared with the same quality attribute from its distorted version. This explains why models proposed by Sprent (1969) and Chan & Mak (1983, 1984) are inappropriate because they expressed a quality attribute measured from reference image as a linear combination of difference quality attributes in the distorted image. On the other hand, simultaneous relationships produce more than one similarity measures; one measure for each image quality attribute. This is clearly not suitable for many image processing problems as Keelan (2002) stated that the image quality measure is a single number correlated with a perceived attribute of quality in an image.

This study presents the model for the multidimensional unreplicated linear functional relationship (MULFR) with single slope and estimates the parameters. The properties of the estimated parameters are investigated. The coefficient of determination for this model, labeled as R_p^2 and its properties are also discussed.

4.1 Formulation of Multidimensional ULFR (MULFR) with Single Slope

We have seen the needs for proposing a new ISM to accommodate the issues mentioned in Chapter 2. We consider the ULFR model in Section 3.3 and derived a preliminary 1-to-1 image comparison measure using coefficient of determination, R_F^2. These results had been published in Chang et al. (2007) and Chang et al. (2008a, 2008b). However, this ULFR model still considered one image attribute. To deal with multiple image quality attributes, we developed and extended the ULFR model to multidimensional observations (MULFR) with single slope. Lastly, the coefficient of determination for the MULFR model and its properties are derived to obtain an overall assessment of the localized quality features.

4.1.1 The MULFR model

Divide the reference image *X* and processed image *Y* into *n* disjoint windows of size $w \times w$. Suppose that $\boldsymbol{Y}_i = \left(Y_{1i}, Y_{2i}, \ldots, Y_{pi}\right)'$ and $\boldsymbol{X}_i = \left(X_{1i}, X_{2i}, \ldots, X_{pi}\right)'$ are two linearly related unobservable true localized image quality attributes

$$\boldsymbol{Y}_i = \boldsymbol{\alpha} + \beta \boldsymbol{X}_i,\ i = 1, 2, \ldots, n \tag{4.1}$$

where Y_{ki} and X_{ki} , $k=1,2,\ldots,p;\ \ i=1,2,\ldots,n$ are p unobservable image quality attributes measured from the n windows of the reference image and distorted image respectively, and $\boldsymbol{\alpha}=\left(\alpha_1,\alpha_2,\ldots,\alpha_p\right)'$ are intercepts and β is the slope of the linear function.

Now, we assume that the two corresponding random vector variables $\boldsymbol{y}_i=\left(y_{1i},y_{2i},\ldots,y_{pi}\right)'$ and $\boldsymbol{x}_i=\left(x_{1i},x_{2i},\ldots,x_{pi}\right)'$ are observed localized image quality attributes measured with errors $\boldsymbol{\delta}_i=\left(\delta_{1i},\delta_{2i},\ldots,\delta_{pi}\right)'$ and $\boldsymbol{\varepsilon}_i=\left(\varepsilon_{1i},\varepsilon_{2i},\ldots,\varepsilon_{pi}\right)'$

$$\left.\begin{aligned}\boldsymbol{x}_i&=\boldsymbol{X}_i+\boldsymbol{\delta}_i\\ \boldsymbol{y}_i&=\boldsymbol{Y}_i+\boldsymbol{\varepsilon}_i\end{aligned}\right\} i=1,2,\ldots,n. \tag{4.2}$$

Assuming both error vectors are mutually and independently normally distributed with

(i) $E\left(\boldsymbol{\delta}_i\right)=\boldsymbol{0}=E\left(\boldsymbol{\varepsilon}_i\right)$

(ii) $Var\left(\delta_{ki}\right)=\sigma^2$ and $Var\left(\varepsilon_{ki}\right)=\tau^2$ for $k=1,2,\ldots,p;\ \ i=1,2,\ldots,n$

(iii) $Cov\left(\delta_{ki},\delta_{kj}\right)=0=Cov\left(\varepsilon_{ki},\varepsilon_{kj}\right)$, for $\forall i\neq j;\ \ i,j=1,2,\ldots,n$

$Cov\left(\delta_{ki},\delta_{hi}\right)=0=Cov\left(\varepsilon_{ki},\varepsilon_{hi}\right), \forall h\neq k;\ \ h,k=1,2,\ldots,p,$

$\forall i;\ \ i=1,2,\ldots,n$

$Cov\left(\delta_{ki},\varepsilon_{hj}\right)=0$ for $\forall i,j;\ \ i,j=1,2,\ldots,n$ and $\forall h,k;\ \ h,k=1,2,\ldots,p$

That is $\boldsymbol{\delta}_i\sim NID\left(\boldsymbol{0},\boldsymbol{\Omega}_{22}\right)$ and $\boldsymbol{\varepsilon}_i\sim NID\left(\boldsymbol{0},\boldsymbol{\Omega}_{11}\right)$ where $\boldsymbol{\Omega}_{11}=\begin{pmatrix}\tau^2&0&0&0\\0&\tau^2&0&0\\0&0&\ddots&0\\0&0&0&\tau^2\end{pmatrix}=\tau^2\boldsymbol{I}$,

$\boldsymbol{\Omega}_{22}=\begin{pmatrix}\sigma^2&0&0&0\\0&\sigma^2&0&0\\0&0&\ddots&0\\0&0&0&\sigma^2\end{pmatrix}=\sigma^2\boldsymbol{I}$ and let $\boldsymbol{v}_i=\begin{pmatrix}\boldsymbol{\varepsilon}_i\\ \boldsymbol{\delta}_i\end{pmatrix}$, then $Cov\left(\boldsymbol{v}_i,\boldsymbol{v}_i\right)=\boldsymbol{\Omega}=\begin{pmatrix}\boldsymbol{\Omega}_{11}&\boldsymbol{\Omega}_{12}\\ \boldsymbol{\Omega}_{21}&\boldsymbol{\Omega}_{22}\end{pmatrix}$ are diagonal variance-covariance matrices, $\boldsymbol{\Omega}_{12}=\boldsymbol{\Omega}_{21}=\boldsymbol{0}$, $\boldsymbol{\Omega}_{11}$ and $\boldsymbol{\Omega}_{22}$ are positive definite.

4.1.2 Estimation of Parameters

We start with the joint probability density function of $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$

$$f(\boldsymbol{x}_i, \boldsymbol{y}_i) = \frac{1}{\left(\sqrt{2\pi}\right)^r |\boldsymbol{\Omega}|^{1/2}} \exp\left[-\frac{1}{2}\left\{\begin{pmatrix} \boldsymbol{y}_i - E(\boldsymbol{y}_i) \\ \boldsymbol{x}_i - E(\boldsymbol{x}_i) \end{pmatrix}' \boldsymbol{\Omega}^{-1} \begin{pmatrix} \boldsymbol{y}_i - E(\boldsymbol{y}_i) \\ \boldsymbol{x}_i - E(\boldsymbol{x}_i) \end{pmatrix}\right\}\right]$$

$$= \frac{1}{(2\pi)^{r/2} |\boldsymbol{\Omega}|^{1/2}} \exp\left[-\frac{1}{2}\left\{\left[(\boldsymbol{y}_i - \boldsymbol{Y}_i)' \quad (\boldsymbol{x}_i - \boldsymbol{X}_i)'\right] \boldsymbol{\Omega}^{-1} \begin{pmatrix} \boldsymbol{y}_i - \boldsymbol{Y}_i \\ \boldsymbol{x}_i - \boldsymbol{X}_i \end{pmatrix}\right\}\right] \qquad (4.3)$$

where $r = 2p$, $E(\boldsymbol{x}_i) = E(\boldsymbol{X}_i + \boldsymbol{\delta}_i) = \boldsymbol{X}_i$ and $E(\boldsymbol{y}_i) = E(\boldsymbol{Y}_i + \boldsymbol{\varepsilon}_i) = \boldsymbol{Y}_i$. The likelihood function for Equation (4.3) is

$$L = \prod_{i=1}^{n} f(\boldsymbol{x}_i, \boldsymbol{y}_i) = \prod_{i=1}^{n} \frac{1}{(2\pi)^{r/2} |\boldsymbol{\Omega}|^{1/2}} \exp\left[-\frac{1}{2}\left\{\left[(\boldsymbol{y}_i - \boldsymbol{Y}_i)' \quad (\boldsymbol{x}_i - \boldsymbol{X}_i)'\right] \boldsymbol{\Omega}^{-1} \begin{pmatrix} \boldsymbol{y}_i - \boldsymbol{Y}_i \\ \boldsymbol{x}_i - \boldsymbol{X}_i \end{pmatrix}\right\}\right]$$

$$= \frac{1}{(2\pi)^{rn/2} |\boldsymbol{\Omega}|^{n/2}} \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{n}\left[(\boldsymbol{y}_i - \boldsymbol{Y}_i)' \quad (\boldsymbol{x}_i - \boldsymbol{X}_i)'\right] \begin{pmatrix} \boldsymbol{\Omega}_{11}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{22}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_i - \boldsymbol{Y}_i \\ \boldsymbol{x}_i - \boldsymbol{X}_i \end{pmatrix}\right\}\right]$$

$$= \frac{1}{(2\pi)^{rn/2} |\boldsymbol{\Omega}|^{n/2}} \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{n}\left((\boldsymbol{y}_i - \boldsymbol{Y}_i)' \boldsymbol{\Omega}_{11}^{-1} (\boldsymbol{y}_i - \boldsymbol{Y}_i) + (\boldsymbol{x}_i - \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{x}_i - \boldsymbol{X}_i)\right)\right\}\right]$$

$$= \frac{1}{\mathrm{K} |\boldsymbol{\Omega}|^{n/2}} \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{n}\left[(\boldsymbol{x}_i - \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{x}_i - \boldsymbol{X}_i) + (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i)' \boldsymbol{\Omega}_{11}^{-1} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i)\right]\right\}\right]$$

where $\mathrm{K} = (2\pi)^{rn/2}$ and the log-likelihood function is

$L^* = \ln L$

$$= -\ln \mathrm{K} - \frac{n}{2}\ln|\boldsymbol{\Omega}| - \frac{1}{2}\sum_{i=1}^{n}\left[(\boldsymbol{x}_i - \boldsymbol{X}_i)' \boldsymbol{\Omega}_{22}^{-1} (\boldsymbol{x}_i - \boldsymbol{X}_i) + (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i)' \boldsymbol{\Omega}_{11}^{-1} (\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i)\right] \qquad (4.4)$$

To guarantee a solution for the problem of maximizing Equation (4.4), an additional assumption proposed by Kendall & Stuart (1979) will be used, namely,

(iv) $\quad \boldsymbol{\Omega}_{11} = \lambda\boldsymbol{\Omega}_{22} \iff \boldsymbol{\Omega}_{11}^{-1} = \frac{1}{\lambda}\boldsymbol{\Omega}_{22}^{-1} \iff \tau^2 = \lambda\sigma^2$ where the ratio of error variances λ is a known constant.

In this case, Equation (4.4) becomes

$$L^* = -\ln \mathrm{K} - \frac{n}{2}\ln \lambda^p \left|\boldsymbol{\Omega}_{22}\right|^2$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left[\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right)'\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)'\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\right]$$

$$= -\ln \mathrm{K} - \frac{n}{2}\ln \lambda^p - n\ln\left|\boldsymbol{\Omega}_{22}\right|$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left[\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right)'\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)'\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\right] \quad (4.5)$$

where $\left|\boldsymbol{\Omega}\right| = \left|\begin{pmatrix}\boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}_{12}\\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22}\end{pmatrix}\right| = \left|\boldsymbol{\Omega}_{11}\boldsymbol{\Omega}_{22}\right| = \left|\lambda\boldsymbol{\Omega}_{22}\boldsymbol{\Omega}_{22}\right| = \lambda^p\left|\boldsymbol{\Omega}_{22}\right|^2$.

There are $(np + p + 2)$ parameters to be estimated, which are $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_p$, $\boldsymbol{\alpha}$, β, and σ^2.

Theorem 4.1 (Vector Derivative): Consider the quadratic form

$$f(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$$

where $\boldsymbol{x}$ is $n\times 1$, $\boldsymbol{A}$ is a square matrix of order n and $\boldsymbol{A}$ does not depend on $\boldsymbol{x}$, then

$$\frac{\partial f}{\partial \boldsymbol{x}} = \left(\boldsymbol{A} + \boldsymbol{A}'\right)\boldsymbol{x}$$

and if $\boldsymbol{A}$ is symmetric, $\frac{\partial f}{\partial \boldsymbol{x}} = 2\boldsymbol{A}\boldsymbol{x}$.

From the vector derivative formula for quadratic matrix equation evaluating to a scalar, we have

$$\frac{\partial L^*}{\partial\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right)} = -\frac{1}{2\lambda}\sum_{i=1}^{n}\left\{\left(\boldsymbol{\Omega}_{22}^{-1}\right) + \left(\boldsymbol{\Omega}_{22}^{-1}\right)'\right\}\left[\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right) + \boldsymbol{\alpha}\right]$$
$$= -\frac{1}{2\lambda}\sum_{i=1}^{n}2\boldsymbol{\Omega}_{22}^{-1}\left[\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right) + \boldsymbol{\alpha}\right] \qquad \left(\because \left(\boldsymbol{\Omega}_{22}^{-1}\right)' = \boldsymbol{\Omega}_{22}^{-1}\right)$$
$$= -\frac{1}{\lambda}\sum_{i=1}^{n}\boldsymbol{\Omega}_{22}^{-1}\left[\beta\boldsymbol{X}_i - \boldsymbol{y}_i + \boldsymbol{\alpha}\right]$$

and the tangent vector to curve $\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right): \mathbb{R} \to \mathbb{R}^n$ is $\frac{\partial\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right)}{\partial\beta} = \boldsymbol{X}_i$.

By using the Chain rule for vector functions, we have

$$\frac{\partial L^*}{\partial\beta} = \frac{\partial L^*}{\partial\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right)'}\frac{\partial\left(\beta\boldsymbol{X}_i - \boldsymbol{y}_i\right)}{\partial\beta}$$

$$=-\frac{1}{\lambda}\sum_{i=1}^{n}\left(\beta\boldsymbol{X}_i-\boldsymbol{y}_i+\boldsymbol{\alpha}\right)'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i$$

$$=\frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i$$

Therefore, differentiate Equation (4.5) with respect to β and set the result equal to zero $\left(\frac{\partial L^*}{\partial\beta}=0\right)$, yields

$$\sum_{i=1}^{n}\frac{1}{\sigma^2}\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)'\boldsymbol{I}\left(\boldsymbol{X}_i\right)=0 \qquad \left(\because\boldsymbol{\Omega}_{22}=\sigma^2\boldsymbol{I}\right)$$

$$\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{X}_i-\boldsymbol{\alpha}'\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\boldsymbol{X}_i-\beta\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\boldsymbol{X}_i'\boldsymbol{X}_i=0$$

$$\therefore\hat{\beta}=\frac{\sum_{i=1}^{n}\boldsymbol{y}_i'\hat{\boldsymbol{X}}_i-\hat{\boldsymbol{\alpha}}'\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\hat{\boldsymbol{X}}_i}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i} \tag{4.6}$$

Similarly, differential Equation (4.5) with respect $\boldsymbol{\alpha}$, $\boldsymbol{X}_i$ and σ give the following results

$$\frac{\partial L^*}{\partial\boldsymbol{X}_i}=-\frac{1}{2}\left\{-2\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{x}_i-\boldsymbol{X}_i\right)+\frac{2}{\lambda}\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)\left(-\beta\right)\right\}=\boldsymbol{0}$$

$$\left(\boldsymbol{x}_i-\boldsymbol{X}_i\right)+\frac{1}{\lambda}\beta\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)=\boldsymbol{0}$$

$$-\left(\lambda+\beta^2\right)\boldsymbol{X}_i+\left(\lambda\boldsymbol{x}_i+\beta\boldsymbol{y}_i-\beta\boldsymbol{\alpha}\right)=\boldsymbol{0}$$

$$\therefore\hat{\boldsymbol{X}}_i=\frac{\lambda\boldsymbol{x}_i+\hat{\beta}\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}\right)}{\lambda+\hat{\beta}^2} \tag{4.7}$$

and,

$$\frac{\partial L^*}{\partial\boldsymbol{\alpha}}=-\frac{1}{2}\sum_{i=1}^{n}\frac{-2}{\lambda}\boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)=\boldsymbol{0}$$

$$\sum_{i=1}^{n}\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)=\boldsymbol{0}$$

$\because\boldsymbol{\Omega}_{22}$ is positive definite and diagonal and $\boldsymbol{\Omega}_{22}=\sigma^2\boldsymbol{I}$

$$\sum_{i=1}^{n}\boldsymbol{y}_i-n\boldsymbol{\alpha}-\beta\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\boldsymbol{X}_i=\boldsymbol{0}$$

$$\therefore \hat{\boldsymbol{\alpha}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{y}_i - \hat{\beta}\frac{1}{n}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\hat{\boldsymbol{X}}_i$$

Substitute Equation (4.7) yields

$$\hat{\boldsymbol{\alpha}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{y}_i - \hat{\beta}\frac{1}{n}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\left[\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right]$$

$$\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{y}} - \frac{1}{\left(\lambda + \hat{\beta}^2\right)}\left(\lambda\hat{\beta}\overline{\boldsymbol{x}} + \hat{\beta}^2\overline{\boldsymbol{y}} - \hat{\beta}^2\hat{\boldsymbol{\alpha}}\right)$$

$$\hat{\boldsymbol{\alpha}} - \frac{\hat{\beta}^2\hat{\boldsymbol{\alpha}}}{\lambda + \hat{\beta}^2} = \left(1 - \frac{\hat{\beta}^2}{\lambda + \hat{\beta}^2}\right)\overline{\boldsymbol{y}} - \frac{\lambda\hat{\beta}\overline{\boldsymbol{x}}}{\lambda + \hat{\beta}^2}$$

Multiply $\left(\lambda + \hat{\beta}^2\right)$ yields

$$\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}} \tag{4.8}$$

where $\overline{\boldsymbol{y}} = \left[\overline{y}_1\ \overline{y}_2 \cdots \overline{y}_p\right]'$ and $\overline{\boldsymbol{x}} = \left[\overline{x}_1\ \overline{x}_2 \cdots \overline{x}_p\right]'$.

Substitute Equations (4.7) and (4.8) into Equation (4.6) yields

$$\therefore \hat{\beta} = \frac{\sum_{i=1}^{n}\boldsymbol{y}_i'\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right) - \hat{\boldsymbol{\alpha}}'\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)}{\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}}\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)'\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)}$$

$$= \frac{\left(\lambda + \hat{\beta}^2\right)\left\{\sum_{i=1}^{n}\left(\lambda\boldsymbol{x}_i'\boldsymbol{y}_i + \hat{\beta}\boldsymbol{y}_i'\boldsymbol{y}_i - \hat{\beta}\hat{\boldsymbol{\alpha}}'\boldsymbol{y}_i\right) - \lambda\hat{\boldsymbol{\alpha}}'\sum_{i=1}^{n}\boldsymbol{x}_i - \hat{\beta}\hat{\boldsymbol{\alpha}}'\sum_{i=1}^{n}\boldsymbol{y}_i + n\hat{\beta}\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}}\right\}}{\sum_{i=1}^{n}\left(\lambda^2\boldsymbol{x}_i'\boldsymbol{x}_i + 2\lambda\hat{\beta}\boldsymbol{x}_i'\boldsymbol{y}_i - 2\lambda\hat{\beta}\hat{\boldsymbol{\alpha}}'\boldsymbol{x}_i - 2\hat{\beta}^2\hat{\boldsymbol{\alpha}}'\boldsymbol{y}_i + \hat{\beta}^2\boldsymbol{y}_i'\boldsymbol{y}_i + \hat{\beta}^2\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}}\right)}$$

$$= \frac{\left(\lambda + \hat{\beta}^2\right)\left\{\lambda\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i + \hat{\beta}\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - 2n\hat{\beta}\hat{\boldsymbol{\alpha}}'\overline{\boldsymbol{y}} - \lambda n\hat{\boldsymbol{\alpha}}'\overline{\boldsymbol{x}} + n\hat{\beta}\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}}\right\}}{\lambda^2\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i + 2\lambda\hat{\beta}\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - 2\lambda\hat{\beta}\hat{\boldsymbol{\alpha}}'\sum_{i=1}^{n}\boldsymbol{x}_i - 2\hat{\beta}^2\hat{\boldsymbol{\alpha}}'\sum_{i=1}^{n}\boldsymbol{y}_i + \hat{\beta}^2\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i + n\hat{\beta}^2\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}}}$$

$$= \frac{\left(\lambda + \hat{\beta}^2\right)\left\{\lambda\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i + \hat{\beta}\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - 2n\hat{\beta}\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{y}} - \lambda n\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{x}} + n\hat{\beta}\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)\right\}}{\lambda^2\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i + 2\lambda\hat{\beta}\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - 2n\lambda\hat{\beta}\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{x}}_i - 2n\hat{\beta}^2\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{y}}_i + \hat{\beta}^2\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i + n\hat{\beta}^2\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)}$$

$$= \frac{\left(\lambda + \hat{\beta}^2\right)\left\{\lambda\left(\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}\right) + \hat{\beta}\left(\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}\right) + \lambda n\hat{\beta}\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}} + n\hat{\beta}^3\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}\right\}}{\lambda^2\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i + 2\lambda\hat{\beta}\left(\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}\right) + \hat{\beta}^2\left(\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}\right) + 2n\lambda\hat{\beta}^2\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}} + n\hat{\beta}^4\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}}$$

$$= \frac{\left(\lambda+\hat{\beta}^2\right)\left\{\lambda S_{xy}+\hat{\beta}S_{yy}+\lambda n\hat{\beta}\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+n\hat{\beta}^3\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}\right\}}{\lambda^2\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i+2\lambda\hat{\beta}S_{xy}+\hat{\beta}^2S_{yy}+2n\lambda\hat{\beta}^2\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+n\hat{\beta}^4\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}}$$

where $S_{xx}=\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i-n\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}$, $S_{yy}=\sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i-n\bar{\boldsymbol{y}}'\bar{\boldsymbol{y}}$ and $S_{xy}=\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i-n\bar{\boldsymbol{x}}'\bar{\boldsymbol{y}}$.

This implies that

$$\lambda^2\hat{\beta}\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i+2\lambda\hat{\beta}^2S_{xy}+\hat{\beta}^3S_{yy}+2n\lambda\hat{\beta}^3\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+n\hat{\beta}^5\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}$$

$$=\lambda^2S_{xy}+\lambda\hat{\beta}S_{yy}+\lambda^2n\hat{\beta}\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+\lambda n\hat{\beta}^3\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+\lambda\hat{\beta}^2S_{xy}+\hat{\beta}^3S_{yy}+\lambda n\hat{\beta}^3\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+n\hat{\beta}^5\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}$$

$$\Rightarrow \lambda^2\hat{\beta}\sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i-\lambda^2n\hat{\beta}\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}+\lambda\hat{\beta}^2S_{xy}-\lambda^2S_{xy}-\lambda\hat{\beta}S_{yy}=0$$

$$\hat{\beta}^2S_{xy}+\hat{\beta}\left(\lambda S_{xx}-S_{yy}\right)-\lambda S_{xy}=0 \tag{4.9}$$

Solving the quadratic Equation (4.9) yields

$$\hat{\beta}=\frac{-\left(\lambda S_{xx}-S_{yy}\right)\pm\sqrt{\left(\lambda S_{xx}-S_{yy}\right)^2+4\lambda S_{xy}^2}}{2S_{xy}}$$

$$=\frac{\left(S_{yy}-\lambda S_{xx}\right)\pm\sqrt{\left(S_{yy}-\lambda S_{xx}\right)^2+4\lambda S_{xy}^2}}{2S_{xy}}$$

$$\therefore \hat{\beta}=\frac{\left(S_{yy}-\lambda S_{xx}\right)+\sqrt{\left(S_{yy}-\lambda S_{xx}\right)^2+4\lambda S_{xy}^2}}{2S_{xy}} \tag{4.10}$$

The positive sign is used in Equation (4.10) because it gives a maximum to the likelihood function in Equation (4.5) as shown below. From the previous result, we have

$$\frac{\partial L^*}{\partial\beta}=\frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i-\boldsymbol{\alpha}-\beta\boldsymbol{X}_i\right)'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i=\frac{1}{\lambda}\left(\sum\boldsymbol{y}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i-\boldsymbol{\alpha}'\sum\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i-\beta\sum\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$$

and the second order derivative yields

$$\frac{\partial^2L^*}{\partial\beta^2}=\frac{-1}{\lambda}\sum\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i=\frac{-1}{\lambda\sigma^2}\sum\boldsymbol{X}_i'\boldsymbol{I}\boldsymbol{X}_i=\frac{-1}{\lambda\sigma^2}\sum\boldsymbol{X}_i'\boldsymbol{X}_i\,. \quad \left(\because \boldsymbol{\Omega}_{22}=\sigma^2\boldsymbol{I}\right)$$

Since $\sum\boldsymbol{X}_i'\boldsymbol{X}_i>0$ (practically $\boldsymbol{X}\neq\boldsymbol{0}$) and $\lambda>0$, this implies that $\frac{\partial^2L^*}{\partial\beta^2}<0$. The $\hat{\beta}$ s are local maximum points. Now, we let

$$\hat{\beta}=\frac{\left(S_{yy}-\lambda S_{xx}\right)\pm\sqrt{\left(S_{yy}-\lambda S_{xx}\right)^2+4\lambda S_{xy}^2}}{2S_{xy}}=\frac{\Delta}{2S_{xy}}$$

Furthermore, Result 6 from Section 4.3 and Section 4.4.2 shown that $\Delta = 2\hat{\beta}S_{xy} \geq 0$ must be non-negative and therefore the positive square root must always be taken.

$$\frac{\partial L^*}{\partial \sigma} = -\frac{2np}{\sigma} + \sigma^{-3}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right)'\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)'\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\right\} = 0$$

$$\frac{2np}{\sigma} = \frac{1}{\sigma^3}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right)'\left(\boldsymbol{x}_i - \boldsymbol{X}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)'\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta\boldsymbol{X}_i\right)\right\}$$

$$\therefore \hat{\sigma}^2 = \frac{1}{2np}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)\right\} \tag{4.11}$$

Since $\hat{\sigma}^2$ is a bias estimator of $\hat{\sigma}^2$ (Kendall & Stuart, 1979), we multiply Equation (4.11) by $\frac{2n}{n-2}$ yields the consistent estimator

$$\therefore \hat{\sigma}^2 = \frac{1}{(n-2)p}\left\{\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right) + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)\right\} \tag{4.12}$$

Substitute Equation (4.7) into Equation (4.12) yields

$$\hat{\sigma}^2 = \frac{1}{(n-2)p}\left\{\begin{array}{l}\displaystyle\sum_{i=1}^{n}\left(\boldsymbol{x}_i - \frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)'\left(\boldsymbol{x}_i - \frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)\\ \displaystyle+\frac{1}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\left(\frac{\lambda\boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)\right)\end{array}\right\}$$

$$= \frac{1}{(n-2)p}\left\{\begin{array}{l}\displaystyle\sum_{i=1}^{n}\left(\frac{\hat{\beta}^2\boldsymbol{x}_i - \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)'\left(\frac{\hat{\beta}^2\boldsymbol{x}_i - \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}\right)\\ \displaystyle+\frac{1}{\lambda}\sum_{i=1}^{n}\left(\frac{\lambda\boldsymbol{y}_i - \lambda\hat{\beta}\boldsymbol{x}_i - \lambda\hat{\boldsymbol{\alpha}}}{\lambda + \hat{\beta}^2}\right)'\left(\frac{\lambda\boldsymbol{y}_i - \lambda\hat{\beta}\boldsymbol{x}_i - \lambda\hat{\boldsymbol{\alpha}}}{\lambda + \hat{\beta}^2}\right)\end{array}\right\}$$

$$= \frac{1}{p(n-2)\left(\lambda + \hat{\beta}^2\right)^2}\left\{\begin{array}{l}\displaystyle\sum_{i=1}^{n}\hat{\beta}\left(\hat{\beta}\boldsymbol{x}_i + \hat{\boldsymbol{\alpha}} - \boldsymbol{y}_i\right)'\hat{\beta}\left(\hat{\beta}\boldsymbol{x}_i + \hat{\boldsymbol{\alpha}} - \boldsymbol{y}_i\right)\\ \displaystyle+\frac{\lambda^2}{\lambda}\sum_{i=1}^{n}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i\right)\end{array}\right\}$$

$$
= \frac{1}{p(n-2)\left(\lambda+\hat{\beta}^2\right)^2}\left\{\begin{array}{l}\sum_{i=1}^{n}\hat{\beta}^2\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right) \\ +\lambda\sum_{i=1}^{n}\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)\end{array}\right\}
$$

$$
= \frac{1}{p(n-2)\left(\lambda+\hat{\beta}^2\right)^2}\left\{\sum_{i=1}^{n}\left(\lambda+\hat{\beta}^2\right)\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)\right\}
$$

$$
= \frac{1}{p(n-2)\left(\lambda+\hat{\beta}^2\right)}\left\{\sum_{i=1}^{n}\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)\right\}
$$

Result 3: Given the Multidimensional ULFR model with single slope defined by Equations (4.1) and (4.2). The maximum likelihood estimators of $\boldsymbol{\alpha}$, β, $\boldsymbol{X}_i$ and σ_k^2 are

$$
\hat{\boldsymbol{\alpha}} = \overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}
$$

$$
\hat{\beta} = \frac{\left(S_{\boldsymbol{yy}}-\lambda S_{\boldsymbol{xx}}\right)+\sqrt{\left(S_{\boldsymbol{yy}}-\lambda S_{\boldsymbol{xx}}\right)^2+4\lambda S_{\boldsymbol{xy}}^2}}{2S_{\boldsymbol{xy}}}
$$

$$
\hat{\boldsymbol{X}}_i = \frac{\lambda\boldsymbol{x}_i+\hat{\beta}\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}\right)}{\lambda+\hat{\beta}^2}
$$

and
$$
\hat{\sigma}^2 = \frac{1}{p(n-2)\left(\lambda+\hat{\beta}^2\right)}\left\{\sum_{i=1}^{n}\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i-\hat{\boldsymbol{\alpha}}-\hat{\beta}\boldsymbol{x}_i\right)\right\}
$$

where λ is the ratio of error variances, and $S_{\boldsymbol{xx}} = \sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{x}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}$, $S_{\boldsymbol{yy}} = \sum_{i=1}^{n}\boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}$ and $S_{\boldsymbol{xy}} = \sum_{i=1}^{n}\boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}$.

4.1.3 Graphical Representation of the MULFR Model with Single Slope

The p-dimensional MULFR model with single slope defined in Equations (4.1) and (4.2) can be re-written as

$$
\begin{pmatrix}x_{1i} & x_{2i} & \cdots & x_{pi}\end{pmatrix}' = \begin{pmatrix}X_{1i} & X_{2i} & \cdots & X_{pi}\end{pmatrix}' + \begin{pmatrix}\delta_{1i} & \delta_{2i} & \cdots & \delta_{pi}\end{pmatrix}'
$$

$$
\begin{pmatrix}y_{1i} & y_{2i} & \cdots & y_{pi}\end{pmatrix}' = \begin{pmatrix}Y_{1i} & Y_{2i} & \cdots & Y_{pi}\end{pmatrix}' + \begin{pmatrix}\varepsilon_{1i} & \varepsilon_{2i} & \cdots & \varepsilon_{pi}\end{pmatrix}'
$$

and $\left(Y_{1i} \quad Y_{2i} \quad \cdots \quad Y_{pi}\right)' = \left(\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_p\right)' + \beta\left(X_{1i} \quad X_{2i} \quad \cdots \quad X_{pi}\right)'$, $i = 1, 2, \ldots, n$.

Let $\hat{\boldsymbol{\alpha}}$, $\hat{\beta}$ and $\hat{\boldsymbol{X}}_i$ be the maximum likelihood estimators for $\boldsymbol{\alpha}$, β and $\boldsymbol{X}_i$, respectively. The relation

$$\left(\hat{Y}_{1i} \quad \hat{Y}_{2i} \quad \cdots \quad \hat{Y}_{pi}\right)' = \left(\hat{\alpha}_1 \quad \hat{\alpha}_2 \quad \cdots \quad \hat{\alpha}_p\right)' + \hat{\beta}\left(\hat{X}_{1i} \quad \hat{X}_{2i} \quad \cdots \quad \hat{X}_{pi}\right)'$$

or

$$\begin{aligned} \hat{Y}_{1i} &= \hat{\alpha}_1 + \hat{\beta}\hat{X}_{1i} \\ \hat{Y}_{2i} &= \hat{\alpha}_2 + \hat{\beta}\hat{X}_{2i} \\ &\vdots \\ \hat{Y}_{pi} &= \hat{\alpha}_p + \hat{\beta}\hat{X}_{pi} \end{aligned}$$

are p linear equations with the same slope but different intercepts. The idea of MULFR with single slope is depicted graphically in Figure 4.1. The colored smooth lines represent the fitted ULFR model for corresponding local data sets (elliptic regions shaded with different colors). These colored smooth lines have different slopes and different intercepts. The red colored doted line represents the fitted MULFR model for global data set, which containing all local data sets. Note that all the local (doted) lines and global line have the same slope.

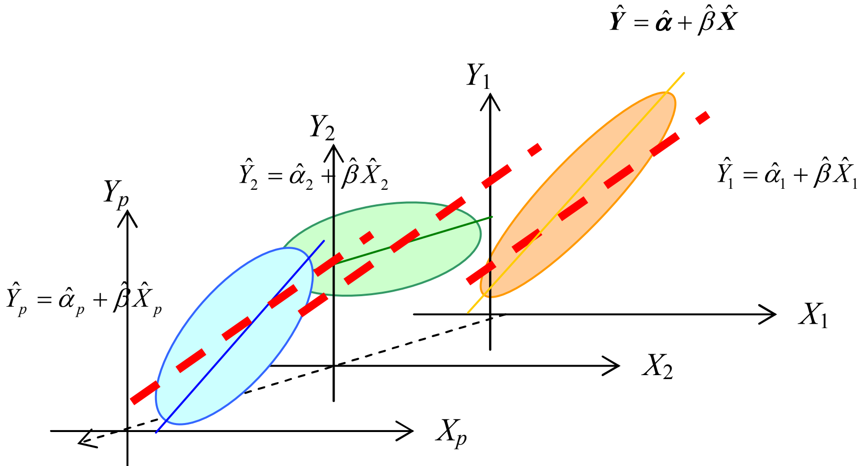

Figure 4.1: Graphical representation of MULFR model. The parallel bolded lines in red color are the fitted model with the same slope with different intercept values.

4.2 Properties of Parameters

Since the probability distributions of the error terms $\boldsymbol{\delta}_i$ and $\boldsymbol{\varepsilon}_i$ in Equation 4.2 is generally unknown, it may be necessary to ensure that $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ to be applied with the conditions that it poses certain good properties such as unbiasness, consistency and asymptotically normal.

4.2.1 Unbiasedness of Parameters

This section discusses the properties of $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$, i.e. the expected values and variance of the estimated parameters. Note that Equation (4.12) can be written as

$$\hat{\beta} = \theta + \sqrt{\theta^2 + \lambda} \qquad \text{where } \theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right) = \frac{S_{\boldsymbol{yy}} - \lambda S_{\boldsymbol{xx}}}{2S_{\boldsymbol{xy}}}.$$

Thus, the expected value of $\hat{\beta}$ is

$$E\left(\hat{\beta}\right) = E\left(\theta + \sqrt{\theta^2 + \lambda}\right) = E\left(\theta\right) + E\left(\sqrt{\theta^2 + \lambda}\right) \tag{4.13}$$

Since Equation (4.13) cannot be solved explicitly, we solve it by using the first order Taylor approximations (or Delta method) (Bain & Engelhardt, 1992) for the mean of $\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right)$. The first expected value in Equation (4.13) can be obtained by the following

$$\begin{aligned}
\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right) &= \theta\left(\boldsymbol{X}_i + \boldsymbol{\delta}_i, \boldsymbol{Y}_i + \boldsymbol{\varepsilon}_i\right) && \text{(from Equation (4.2))} \\
&\doteq \theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right) + \boldsymbol{\delta}_i' \left.\frac{\partial \theta}{\partial \boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i} + \boldsymbol{\varepsilon}_i' \left.\frac{\partial \theta}{\partial \boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i} \\
&= \theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right) + \boldsymbol{\delta}_i' \left.\theta_{\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i} + \boldsymbol{\varepsilon}_i' \left.\theta_{\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i} && (4.14)
\end{aligned}$$

where the partial derivatives are evaluated at the mean $\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right)$. Equation (4.14) is valid only if the error variances, $\sigma_{\boldsymbol{\delta}}^2$ and $\sigma_{\boldsymbol{\varepsilon}}^2$ are small. Since

$$E\left(\boldsymbol{\delta}_i' \left.\theta_{\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i}\right) = E\left[\sum_{k=1}^{p} \delta_{ik} \left.\theta_{x_{ik}}\right|_{x_{ik} = X_{ik}}\right] = \sum_{k=1}^{p} \left.\theta_{x_{ik}}\right|_{x_{ik} = X_{ik}} E\left(\delta_{ik}\right) = 0$$

$$\left(\because E\left(\boldsymbol{\delta}_i\right) = \mathbf{0} \Rightarrow E\left(\delta_{ik}\right) = 0\right)$$

Similarly, we have $E\left(\boldsymbol{\varepsilon}_i' \left.\theta_{\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i}\right) = 0$. Therefore, Equation (4.14) becomes

$$E\left[\theta\left(\boldsymbol{x}_i, \boldsymbol{y}_i\right)\right] \doteq E\left[\theta\left(\boldsymbol{X}_i, \boldsymbol{Y}_i\right)\right] + E\left(\boldsymbol{\delta}_i' \left.\theta_{\boldsymbol{x}_i}\right|_{\boldsymbol{x}_i = \boldsymbol{X}_i}\right) + E\left(\boldsymbol{\varepsilon}_i' \left.\theta_{\boldsymbol{y}_i}\right|_{\boldsymbol{y}_i = \boldsymbol{Y}_i}\right)$$

$$=\theta\left(\boldsymbol{X}_i,\boldsymbol{Y}_i\right)=\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}} \tag{4.15}$$

where $S_{YY}=\sum \boldsymbol{Y}_i'\boldsymbol{Y}_i-n\overline{\boldsymbol{Y}}'\overline{\boldsymbol{Y}}$, $S_{XX}=\sum \boldsymbol{X}_i'\boldsymbol{X}_i-n\overline{\boldsymbol{X}}'\overline{\boldsymbol{X}}$ and $S_{XY}=\sum \boldsymbol{X}_i'\boldsymbol{Y}_i-n\overline{\boldsymbol{X}}'\overline{\boldsymbol{Y}}$.

Now let $\varphi\left(\boldsymbol{x}_i,\boldsymbol{y}_i\right)=\sqrt{\theta^2\left(\boldsymbol{x}_i,\boldsymbol{y}_i\right)+\lambda}$. This implies that $\dfrac{\partial\varphi}{\partial\boldsymbol{x}_i}=\left(\theta^2+\lambda\right)^{-1/2}\theta\dfrac{\partial\theta}{\partial\boldsymbol{x}_i}$.

We have

$$\varphi\left(\boldsymbol{x}_i,\boldsymbol{y}_i\right)=\varphi\left(\boldsymbol{X}_i+\boldsymbol{\delta}_i,\boldsymbol{Y}_i+\boldsymbol{\varepsilon}_i\right)$$

$$\doteq\theta\left(\boldsymbol{X}_i,\boldsymbol{Y}_i\right)+\boldsymbol{\delta}_i'\left.\frac{\partial\varphi}{\partial\boldsymbol{x}_i}\right|_{x_i=X_i}+\boldsymbol{\varepsilon}_i'\left.\frac{\partial\varphi}{\partial\boldsymbol{y}_i}\right|_{y_i=Y_i}$$

$$=\theta\left(\boldsymbol{X}_i,\boldsymbol{Y}_i\right)+\left(\theta^2+\lambda\right)^{-1/2}\theta\boldsymbol{\delta}_i'\left.\theta_{x_i}\right|_{x_i=X_i}+\left(\theta^2+\lambda\right)^{-1/2}\theta\boldsymbol{\varepsilon}_i'\left.\theta_{y_i}\right|_{y_i=Y_i}$$

Hence, the second expected value in Equation (4.13) is

$$E\left[\varphi\left(\boldsymbol{x}_i,\boldsymbol{y}_i\right)\right]\doteq E\left[\varphi\left(\boldsymbol{X}_i,\boldsymbol{Y}_i\right)\right]+\theta\left(\theta^2+\lambda\right)^{-1/2}\mathrm{E}\left(\boldsymbol{\delta}_i'\left.\theta_{x_i}\right|_{x_i=X_i}\right)+\theta\left(\theta^2+\lambda\right)^{-1/2}\mathrm{E}\left(\boldsymbol{\varepsilon}_i'\left.\theta_{y_i}\right|_{y_i=Y_i}\right)$$

$$=\varphi\left(\boldsymbol{X}_i,\boldsymbol{Y}_i\right)+\theta\left(\theta^2+\lambda\right)^{-1/2}(0)+\theta\left(\theta^2+\lambda\right)^{-1/2}(0)$$

$$=\sqrt{\theta^2\left(\boldsymbol{X}_i,\boldsymbol{Y}_i\right)+\lambda}$$

$$=\sqrt{\left(\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}}\right)^2+\lambda} \tag{4.16}$$

From the Equations (4.15) and (4.16), hence Equation (4.13) becomes

$$E\left(\hat{\beta}\right)\doteq\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}}+\sqrt{\left(\frac{S_{YY}-\lambda S_{XX}}{2S_{XY}}\right)^2+\lambda}$$

$$=\frac{\left(S_{YY}-\lambda S_{XX}\right)+\sqrt{\left(S_{YY}-\lambda S_{XX}\right)^2+4\lambda S_{XY}^2}}{2S_{XY}} \tag{4.17}$$

The next step is to show that $S_{XY}=\beta S_{XX}$ and $S_{YY}=\beta^2 S_{XX}=\beta S_{XY}$ as also stated in Lindley (1947).

$$S_{XY}=\sum \boldsymbol{X}_i'\boldsymbol{Y}_i-n\overline{\boldsymbol{X}}'\overline{\boldsymbol{Y}}$$

$$=\sum \boldsymbol{X}_i'\left(\boldsymbol{\alpha}+\beta\boldsymbol{X}_i\right)-n\overline{\boldsymbol{X}}'\left(\boldsymbol{\alpha}+\beta\overline{\boldsymbol{X}}\right)$$

$$=\sum\left(\boldsymbol{X}_i'\boldsymbol{\alpha}+\beta\boldsymbol{X}_i'\boldsymbol{X}_i\right)-n\overline{\boldsymbol{X}}'\boldsymbol{\alpha}-n\beta\overline{\boldsymbol{X}}'\overline{\boldsymbol{X}}$$

$$=\boldsymbol{\alpha}'\sum\boldsymbol{X}_i+\beta\sum\boldsymbol{X}_i'\boldsymbol{X}_i-n\boldsymbol{\alpha}'\overline{\boldsymbol{X}}-n\beta\overline{\boldsymbol{X}}'\overline{\boldsymbol{X}}$$

$$=\boldsymbol{\alpha}'\left(\sum\boldsymbol{X}_i-n\overline{\boldsymbol{X}}\right)+\beta\left(\sum\boldsymbol{X}_i'\boldsymbol{X}_i-n\overline{\boldsymbol{X}}'\overline{\boldsymbol{X}}\right)$$

$$= \beta S_{XX}$$

and $$S_{YY} = \sum \boldsymbol{Y}_i'\boldsymbol{Y}_i - n\bar{\boldsymbol{Y}}'\bar{\boldsymbol{Y}}$$

$$= \sum (\boldsymbol{\alpha} + \beta \boldsymbol{X}_i)'(\boldsymbol{\alpha} + \beta \boldsymbol{X}_i) - n(\boldsymbol{\alpha} + \beta \bar{\boldsymbol{X}})'(\boldsymbol{\alpha} + \beta \bar{\boldsymbol{X}})$$

$$= \sum \left(\boldsymbol{\alpha}'\boldsymbol{\alpha} + 2\beta \boldsymbol{\alpha}'\boldsymbol{X}_i + \beta^2 \boldsymbol{X}_i'\boldsymbol{X}_i\right) - n\left(\boldsymbol{\alpha}'\boldsymbol{\alpha} + 2\beta \boldsymbol{\alpha}'\bar{\boldsymbol{X}} + \beta^2 \bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$

$$= n\boldsymbol{\alpha}'\boldsymbol{\alpha} + 2\beta \boldsymbol{\alpha}' \sum \boldsymbol{X}_i + \beta^2 \sum \boldsymbol{X}_i'\boldsymbol{X}_i - n\left(\boldsymbol{\alpha}'\boldsymbol{\alpha} + 2\beta \boldsymbol{\alpha}'\bar{\boldsymbol{X}} + \beta^2 \bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$

$$= \beta^2 \sum \boldsymbol{X}_i'\boldsymbol{X}_i - n\beta^2 \bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}$$

$$= \beta^2 \left(\sum \boldsymbol{X}_i'\boldsymbol{X}_i - n\bar{\boldsymbol{X}}'\bar{\boldsymbol{X}}\right)$$

$$= \beta^2 S_{XX} = \beta S_{XY}$$

Therefore, Equation (4.17) can be reduced to

$$E\left(\hat{\beta}\right) \doteq \frac{\left(\beta^2 S_{XX} - \lambda S_{XX}\right) + \sqrt{\left(\beta^2 S_{XX} - \lambda S_{XX}\right)^2 + 4\lambda\beta^2 S_{xx}^2}}{2\beta S_{XX}}$$

$$= \frac{\left(\beta^2 - \lambda\right) S_{XX} + \sqrt{\beta^4 S_{xx}^2 - 2\lambda\beta^2 S_{xx}^2 + \lambda^2 S_{xx}^2 + 4\lambda\beta^2 S_{xx}^2}}{2\beta S_{XX}}$$

$$= \frac{\left(\beta^2 - \lambda\right) S_{XX} + \sqrt{\beta^4 S_{xx}^2 + 2\lambda\beta^2 S_{xx}^2 + \lambda^2 S_{xx}^2}}{2\beta S_{XX}}$$

$$= \frac{\left(\beta^2 - \lambda\right) S_{XX} + \sqrt{\left(\beta^2 S_{XX} + \lambda S_{XX}\right)^2}}{2\beta S_{XX}}$$

$$= \frac{\left(\beta^2 - \lambda\right) S_{XX} + \left(\beta^2 S_{XX} + \lambda S_{XX}\right)}{2\beta S_{XX}}$$

$$= \frac{\left(\beta^2 - \lambda\right) + \left(\beta^2 + \lambda\right)}{2\beta}$$

$$= \frac{2\beta^2}{2\beta} = \beta$$

From the Equation (4.7), we have

$$\hat{\boldsymbol{\alpha}} = \bar{\boldsymbol{y}} - \hat{\beta}\bar{\boldsymbol{x}}$$

$$\Rightarrow E\left(\hat{\boldsymbol{\alpha}}\right) = E\left(\bar{\boldsymbol{y}} - \hat{\beta}\bar{\boldsymbol{x}}\right) = \bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}E\left(\hat{\beta}\right) \doteq \bar{\boldsymbol{y}} - \beta\bar{\boldsymbol{x}} = \boldsymbol{\alpha}$$

Result 4: Given the MULFR model stated in Equations (4.1) and (4.2), then the maximum likelihood estimators of $\boldsymbol{\alpha}$ and β are approximate unbiased estimators, i.e.

$$E\left(\hat{\beta}\right) \doteq \beta \text{ and } E\left(\hat{\boldsymbol{\alpha}}\right) \doteq \boldsymbol{\alpha}$$

4.2.2 Variance and Covariance of the Expected Parameters

To find the variance of the $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$, we consider the Fisher Information Matrix (Klein & Neudecker, 2000) of parameters $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$. The first order partial derivatives for log-likelihood function are given by

$$\frac{\partial L^*}{\partial \boldsymbol{\alpha}} = \frac{1}{\lambda}\sum \boldsymbol{\Omega}_{22}^{-1}\left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i\right)$$

and
$$\frac{\partial L^*}{\partial \beta} = \frac{1}{\lambda}\sum \left(\boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{X}_i\right)' \boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i$$

The second order partial derivatives for log-likelihood function and their negative expected values are given by

$$\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}'} = -\frac{n}{\lambda}\boldsymbol{\Omega}_{22}^{-1}, \qquad \text{hence } E\left(-\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}\partial \boldsymbol{\alpha}'}\right) = \frac{n}{\lambda}\boldsymbol{\Omega}_{22}^{-1}$$

$$\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}'\partial \beta} = -\frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\right), \qquad \text{hence } E\left(-\frac{\partial^2 L^*}{\partial \boldsymbol{\alpha}'\partial \beta}\right) = \frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\right)$$

$$\frac{\partial^2 L^*}{\partial \beta^2} = -\frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right), \qquad \text{hence } E\left(-\frac{\partial^2 L^*}{\partial \beta^2}\right) = \frac{1}{\lambda}\sum\left(\boldsymbol{X}_i'\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$$

and
$$\frac{\partial^2 L^*}{\partial \beta\partial \boldsymbol{\alpha}} = -\frac{1}{\lambda}\sum\left(\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right), \qquad \text{hence } E\left(-\frac{\partial^2 L^*}{\partial \beta\partial \boldsymbol{\alpha}}\right) = \frac{1}{\lambda}\sum\left(\boldsymbol{\Omega}_{22}^{-1}\boldsymbol{X}_i\right)$$

Next, we find the estimated FIM for $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ given by

$$F = \begin{bmatrix} \frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1} & \frac{1}{\lambda}\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right) \\ \frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right) & \frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is a $p \times p$ matrix given by $\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}$,

B is a $p \times 1$ matrix given by $\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)$,

C is a $1 \times p$ matrix given by $\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)$ and $C' = B$,

D is a 1×1 matrix given by $\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)$.

Thus, the inverse of F (Zhang, 1999) is

$$F^{-1}=\begin{bmatrix}\left(A-BD^{-1}C\right)^{-1} & -A^{-1}B\left(D-CA^{-1}B\right)^{-1}\\ -D^{-1}C\left(A-BD^{-1}C\right)^{-1} & \left(D-CA^{-1}B\right)^{-1}\end{bmatrix}$$

Therefore, we obtained the following results:

$$V\hat{a}r\left(\hat{\beta}\right)=\left(D-CA^{-1}B\right)^{-1}$$

$$=\left[\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)-\frac{1}{\lambda}\left\{\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\right\}\left(\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)^{-1}\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right]^{-1}$$

$$=\lambda\left[\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)-\frac{1}{n}\left\{\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\right\}\left(\hat{\boldsymbol{\Omega}}_{22}\right)\left\{\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}\right]^{-1}$$

$$=\lambda\hat{\sigma}^2\left[\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i-\frac{1}{n}\left(\sum\hat{\boldsymbol{X}}_i'\right)\left(\sum\hat{\boldsymbol{X}}_i\right)\right]^{-1}\qquad\left(\because\boldsymbol{\Omega}_{22}=\sigma^2\boldsymbol{I}\right)\tag{4.18}$$

$$V\hat{a}r\left(\hat{\boldsymbol{\alpha}}\right)=\left(A-BD^{-1}C\right)^{-1}$$

$$=\left[\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}-\frac{1}{\lambda}\left\{\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}\left\{\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}^{-1}\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\right]^{-1}$$

$$=\left[\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}-\frac{1}{\lambda}\left\{\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}\left\{\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}^{-1}\left\{\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\right\}\right]^{-1}$$

$$=\lambda\left[\frac{n}{\hat{\sigma}^2}\boldsymbol{I}-\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i\right\}\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}\qquad\left(\because\boldsymbol{\Omega}_{22}=\sigma^2\boldsymbol{I}\right)$$

$$=\lambda\hat{\sigma}^2\left[n\boldsymbol{I}-\left\{\sum\hat{\boldsymbol{X}}_i\right\}\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}\tag{4.19}$$

and $C\hat{o}v\left(\hat{\boldsymbol{\alpha}},\hat{\beta}\right)=-D^{-1}C\left(A-BD^{-1}C\right)^{-1}$

$$=-\left\{\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}^{-1}\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\left[\frac{n}{\lambda}\hat{\boldsymbol{\Omega}}_{22}^{-1}-\frac{1}{\lambda}\left\{\sum\left(\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}\left\{\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\hat{\boldsymbol{X}}_i\right)\right\}^{-1}\frac{1}{\lambda}\sum\left(\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{\Omega}}_{22}^{-1}\right)\right]^{-1}$$

$$=-\lambda\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left(\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\right)\left[\frac{n}{\hat{\sigma}^2}\boldsymbol{I}-\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i\right\}\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\frac{1}{\hat{\sigma}^2}\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}$$

$$=-\lambda\hat{\sigma}^2\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left(\sum\hat{\boldsymbol{X}}_i'\right)\left[n\boldsymbol{I}-\left\{\sum\hat{\boldsymbol{X}}_i\right\}\left\{\sum\hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum\hat{\boldsymbol{X}}_i'\right\}\right]^{-1}\tag{4.20}$$

where $\hat{\boldsymbol{\Omega}}_{22} = \hat{\sigma}^2 \boldsymbol{I}$ and $\hat{\boldsymbol{X}}_i = \dfrac{\lambda \boldsymbol{x}_i + \hat{\beta}\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}}\right)}{\lambda + \hat{\beta}^2}$.

Result 5: Given that $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are MLE of $\boldsymbol{\alpha}$ and β, respectively for the MULFR model, then

$$V\hat{a}r\left(\hat{\beta}\right) = \lambda\hat{\sigma}^2\left[\sum \hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i - \frac{1}{n}\left(\sum \hat{\boldsymbol{X}}_i'\right)\left(\sum \hat{\boldsymbol{X}}_i\right)\right]^{-1}$$

$$V\hat{a}r\left(\hat{\boldsymbol{\alpha}}\right) = \lambda\hat{\sigma}^2\left[n\boldsymbol{I} - \left\{\sum \hat{\boldsymbol{X}}_i\right\}\left\{\sum \hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum \hat{\boldsymbol{X}}_i'\right\}\right]^{-1}$$

$$C\hat{o}v\left(\hat{\boldsymbol{\alpha}}, \hat{\beta}\right) = -\lambda\hat{\sigma}^2\left\{\sum \hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left(\sum \hat{\boldsymbol{X}}_i'\right)\left[n\boldsymbol{I} - \left\{\sum \hat{\boldsymbol{X}}_i\right\}\left\{\sum \hat{\boldsymbol{X}}_i'\hat{\boldsymbol{X}}_i\right\}^{-1}\left\{\sum \hat{\boldsymbol{X}}_i'\right\}\right]^{-1}$$

4.2.3 Consistent Estimators

Definition 4.1: An estimator $\hat{\theta}_n$ of θ based on a random sample of size *n* is a consistent estimator of θ if $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| > \omega\right) = 0$ for every $\omega > 0$.

Theorem 4.2 (Chebyshev's Inequality): Let *X* be a continuous random variable with finite mean $\mu = E(X)$ and variance $\sigma^2 = V(X)$, and let $\varepsilon > 0$ be any positive real number. Then

$$P\left(|X - \mu| \ge \varepsilon\right) \le \frac{V(X)}{\varepsilon^2}$$

Theorem 4.3: An unbiased estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if $\lim_{n\to\infty} Var\left(\hat{\theta}_n\right) = 0$.

It has been shown in Result 4 that estimators $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are approximately unbiased. From Chebyshev's inequality, we see that

$$P\left(\left|\hat{\beta}-\beta\right| \geq \omega\right) \leq \frac{Var\left(\hat{\beta}\right)}{\omega^2} \tag{4.21}$$

Without loss of generality, we remove the equality inside the probability in Equation (4.21) and combined with Definition 1, yields

$$\lim_{n\to\infty} P\left(\left|\hat{\beta}-\beta\right| > \omega\right) \leq \frac{1}{\omega^2} \lim_{n\to\infty} Var\left(\hat{\beta}\right) = 0$$

$$\Rightarrow \lim_{n\to\infty} Var\left(\hat{\beta}\right) = 0 \qquad \text{for every } \omega > 0 .$$

In order to show that the estimator $\hat{\beta}$ is consistent, we need to indicate $Var\left(\hat{\beta}\right) \to 0$ as $n \to \infty$. This can be obtained from the results in Result 5 as follows

$$\lim_{n\to\infty} V\hat{a}r\left(\hat{\beta}\right) = \lambda \lim_{n\to\infty} \left[\frac{\hat{\sigma}^2}{\sum \hat{\boldsymbol{X}}_i' \hat{\boldsymbol{X}}_i - \frac{1}{n}\left(\sum \hat{\boldsymbol{X}}_i'\right)\left(\sum \hat{\boldsymbol{X}}_i\right)} \right]$$

$$= \lambda \lim_{n\to\infty} \left[\frac{\frac{1}{n-2}\left\{ \sum_{i=1}^{n} \left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right)' \left(\boldsymbol{x}_i - \hat{\boldsymbol{X}}_i\right) + \frac{1}{\lambda} \sum_{i=1}^{n} \left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right)' \left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\hat{\boldsymbol{X}}_i\right) \right\}}{\sum \hat{\boldsymbol{X}}_i' \hat{\boldsymbol{X}}_i - \frac{1}{n}\left(\sum \hat{\boldsymbol{X}}_i'\right)\left(\sum \hat{\boldsymbol{X}}_i\right)} \right]$$

$$= \lambda \frac{0}{\sum \hat{\boldsymbol{X}}_i' \hat{\boldsymbol{X}}_i} = 0$$

Similarly, we can show $V\hat{a}r\left(\hat{\boldsymbol{\alpha}}\right) \to 0$ as $n \to \infty$.

4.2.4 Asymptotic Normality and Efficiency

Theorem 4.4 (Multivariate Central Limit Theorem for iid sequences): If $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are independent and identically distributed with mean $\boldsymbol{\mu} \in R^k$ and covariance Σ, where Σ has finite entries, then

$$\sqrt{n}\left(\bar{\boldsymbol{X}}_n - \boldsymbol{\mu}\right) \xrightarrow{d} N_k\left(\mathbf{0}, \Sigma\right)$$

Theorem 4.5 (Weak Law of Large Number): Let $X_1, X_2, \ldots, X_n$ be an independent trials process with a continuous density function *f*, finite expected value μ, and finite variance σ^2. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the sum of the X_i. Then for any real number $\varepsilon > 0$ we have

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) = 0$$

or equivalently, $\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| < \varepsilon\right) = 1$.

Theorem 4.6 (Taylor's Theorem for Multivariate Functions – Linear Form): Suppose $X \subseteq \boldsymbol{R}^n$ is open, $x \in X$, and $f : X \to \boldsymbol{R}^m$ is differentiable. Then

$$f\left(x+h\right) = f\left(x\right) + Df\left(x\right)\left(h\right) + o\left(\left|h\right|\right) \text{ as } h \to 0$$

where $Df\left(x\right) = \frac{\partial f}{\partial x}$ and $o\left(\bullet\right)$ is an error term.

To prove the asymptotic normality and efficiency of $\hat{\beta}$, we shall assume that the first two derivatives of the log-likelihood function L^* (see Equation 4.4) exist, that $E\left(\frac{\partial L^*}{\partial \beta}\right) = 0$ and $I\left(\beta\right) = -E\left(\frac{\partial^2 L^*}{\partial \beta^2}\right) = E\left\{\left(\frac{\partial L^*}{\partial \beta}\right)^2\right\}$ is the Fisher Information, where $I\left(\beta\right) > 0$.

Using Taylor's Theorem, we have

$$\left(\frac{\partial L^*}{\partial \beta}\right)_{\hat{\beta}} = \left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0} + \left(\hat{\beta} - \beta_0\right)\left(\frac{\partial^2 L^*}{\partial \beta^2}\right)_{\beta^*} \tag{4.22}$$

where β^* is some value between $\hat{\beta}$ and β_0. Since $\hat{\beta}$ is the maximum likelihood estimator of L^*, this implies that the left-hand side of (4.22) is zero, $\left(\frac{\partial L^*}{\partial \beta}\right)_{\hat{\beta}} = 0$. On its right-hand side, both $\frac{\partial L^*}{\partial \beta}$ and $\frac{\partial^2 L^*}{\partial \beta^2}$ are sums of independent identical variates, and as $n \to \infty$ each therefore converges to its expectation by the Weak Law of Large Number. The first of these expectations is zero by $E\left(\frac{\partial L^*}{\partial \beta}\right) = 0$ and the second non-zero by $I(\beta)$. Since the right-hand side of (4.22) as a whole must converge to zero, to remain equal to the left, we see that we must have $\left(\hat{\beta} - \beta_0\right)$ converging to zero as $n \to \infty$, so that $\hat{\beta}$ is a consistent estimator under our assumptions.

We now re-write (4.22) in the form

$$\left(\hat{\beta} - \beta_0\right) = -\frac{\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}}{\left(\frac{\partial^2 L^*}{\partial \beta^2}\right)_{\beta^*}} \quad \text{or} \quad \sqrt{n}\left(\hat{\beta} - \beta_0\right)\sqrt{I(\beta_0)} = \frac{\sqrt{n}\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0} \Big/ \sqrt{I(\beta_0)}}{\left(\frac{\partial^2 L^*}{\partial \beta^2}\right)_{\beta^*} \Big/ \left\{-I(\beta_0)\right\}} \tag{4.23}$$

In the denominator on the right of (4.23) we have, since $\hat{\beta}$ is consistent for β_0 and β^* lies between them, from

$$\frac{1}{n}\left[\frac{\partial^2 L^*}{\partial \beta^2}\right]_{\beta=\hat{\beta}} \xrightarrow[n\to\infty]{} \frac{1}{n}\left[\frac{\partial^2 L^*}{\partial \beta^2}\right]_{\beta=\beta_0}$$

$$\lim_{n\to\infty} P\left\{\left[\frac{\partial^2 L^*}{\partial \beta^2}\right]_{\beta=\hat{\beta}} = E_0\left[\frac{\partial^2 L^*}{\partial \beta^2}\right]_{\beta=\beta_0}\right\} = 1$$

and $I(\beta) = -E\left(\frac{\partial^2 L^*}{\partial \beta^2}\right) = E\left\{\left(\frac{\partial L^*}{\partial \beta}\right)^2\right\}$, we have

$$\lim_{n\to\infty} P\left\{\left[\frac{\partial^2 L^*}{\partial \beta^2}\right]_{\beta^*} = -I(\beta_0)\right\} = 1 \qquad \because \beta^* \in \left[\hat{\beta}, \beta_0\right] \tag{4.24}$$

So that the denominator converges to unity. The numerator on the right of (4.23) is the ratio to $\sqrt{I(\beta_0)}$ of the sum of the n independent identical variates. This sum has zero mean by $E\left[\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}\right]=0$ and variance

$$Var\left[\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}\right]=E\left[\left(\frac{\partial L^*}{\partial \beta}\right)^2_{\beta_0}\right]-\left\{E\left[\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}\right]\right\}^2=I(\beta_0)-0=I(\beta_0).$$

The Central Limit Theorem therefore applies, and the numerator is asymptotically a standard normal variate;

$$\frac{\sqrt{n}\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}}{\sqrt{I(\beta_0)}}=\frac{\sqrt{n}}{\sqrt{I(\beta_0)}}\left[\frac{1}{n}\sum_i^n\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}-0\right]=\frac{\sqrt{n}}{\sqrt{I(\beta_0)}}\left[\frac{1}{n}\sum_i^n\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}-E\left[\left(\frac{\partial L^*}{\partial \beta}\right)_{\beta_0}\right]\right]$$

$$\to N(0,1).$$

The same is therefore true of the right-hand side as a whole. Thus the left-hand side of (4.23) is asymptotically standard normal or in other words,

$$\sqrt{nI(\beta_0)}\left(\hat{\beta}-\beta_0\right)\to N(0,1) \text{ or } \sqrt{n}\left(\hat{\beta}-\beta_0\right)\to N\left(0,\frac{1}{I(\beta_0)}\right).$$

Similar argument applies to prove the asymptotic normality of $\hat{\alpha}_k$, and generalized to $\hat{\boldsymbol{\alpha}}=\left[\hat{\alpha}_1 \quad \hat{\alpha}_2 \quad \cdots \quad \hat{\alpha}_p\right]'$. The $\hat{\alpha}_k$ is asymptotically normally distributed

$$\sqrt{n}\left(\hat{\alpha}_k-\alpha_{k0}\right)\to N\left(0,\frac{1}{I(\alpha_{k0})}\right).$$

4.2.5 Interval Estimation for α and β

For a p-dimensional ULFR model defined in Equations (4.1) and (4.2), we have $2np$-independent observations that are available to estimate $(np+p+2)$-parameters of the population, i.e. p parameters from $\boldsymbol{\alpha}$ and one from β. Hence, the number of degrees of freedom is $np-(p+2)$.

Now, we can define the $(1-a)100\%$ confidence intervals for $\boldsymbol{\alpha}$ and β as

$$\hat{\alpha}_k - Z_{\frac{a}{2}} se(\hat{\alpha}_k) \le \alpha_k \le \hat{\alpha}_k + Z_{\frac{a}{2}} se(\hat{\alpha}_k) \tag{4.25}$$

and

$$\hat{\beta} - Z_{\frac{a}{2}} se(\hat{\beta}) \le \beta \le \hat{\beta} + Z_{\frac{a}{2}} se(\hat{\beta}) \tag{4.26}$$

where a is the level of significance, the standard errors $se(\hat{\alpha}_k)=\sqrt{V\hat{a}r(\hat{\alpha}_k)}$ and $se(\hat{\beta})=\sqrt{V\hat{a}r(\hat{\beta})}$ can be obtained from Result 5.

4.3 Coefficient of Determination for MULFR Model

Re-write the Equations (4.1) and (4.2) as

$$\boldsymbol{y}_i = \boldsymbol{\alpha} + \beta \boldsymbol{X}_i + \boldsymbol{\varepsilon}_i = \boldsymbol{\alpha} + \beta \boldsymbol{x}_i + (\boldsymbol{\varepsilon}_i - \beta \boldsymbol{\delta}_i) = \boldsymbol{\alpha} + \beta \boldsymbol{x}_i + \boldsymbol{V}_i \tag{4.27}$$

where the errors of the model is

$$\boldsymbol{V}_i = \boldsymbol{\varepsilon}_i - \beta \boldsymbol{\delta}_i = \boldsymbol{y}_i - \boldsymbol{\alpha} - \beta \boldsymbol{x}_i,\ i=1,2,\ldots,n \tag{4.28}$$

If $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ are estimators of $\boldsymbol{\alpha}$ and β, respectively, then from the idea of least square estimation and Equation (4.28) we have

$$\hat{\boldsymbol{V}}_i = \boldsymbol{y}_i - \hat{\boldsymbol{y}}_i = \boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i,\ i=1,2,\ldots,n$$

is the residual of the model.

Note from Result 3 that $\hat{\sigma}^2 = \dfrac{SS_E}{p(n-2)} = cSS_E$, where $c=\dfrac{1}{p(n-2)}$ is a constant and define the residual sum of squares as

$$SS_E = \frac{1}{\lambda+\hat{\beta}^2}\sum\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i\right)'\left(\boldsymbol{y}_i - \hat{\boldsymbol{\alpha}} - \hat{\beta}\boldsymbol{x}_i\right) = \frac{1}{\lambda+\hat{\beta}^2}\sum \hat{\boldsymbol{V}}_i'\hat{\boldsymbol{V}}_i$$

$$= \frac{1}{\lambda+\hat{\beta}^2}\left(\sum \boldsymbol{y}_i'\boldsymbol{y}_i - 2\hat{\boldsymbol{\alpha}}'\sum \boldsymbol{y}_i - 2\hat{\beta}\sum \boldsymbol{x}_i'\boldsymbol{y}_i + 2\hat{\beta}\hat{\boldsymbol{\alpha}}'\sum \boldsymbol{x}_i + n\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\alpha}} + \hat{\beta}^2\sum \boldsymbol{x}_i'\boldsymbol{x}_i\right)$$

$$= \frac{1}{\lambda+\hat{\beta}^2}\Bigg(\sum \boldsymbol{y}_i'\boldsymbol{y}_i - 2n\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{y}} - 2\hat{\beta}\sum \boldsymbol{x}_i'\boldsymbol{y}_i + 2n\hat{\beta}\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\overline{\boldsymbol{x}}$$

$$+n\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right)'\left(\overline{\boldsymbol{y}} - \hat{\beta}\overline{\boldsymbol{x}}\right) + \hat{\beta}^2\sum \boldsymbol{x}_i'\boldsymbol{x}_i\Bigg)$$

$$= \frac{1}{\lambda + \hat{\beta}^2}\left(\left[\sum \boldsymbol{y}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{y}}'\overline{\boldsymbol{y}}\right] - 2\hat{\beta}\left[\sum \boldsymbol{x}_i'\boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}\right] + \hat{\beta}^2\left[\sum \boldsymbol{x}_i'\boldsymbol{x}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{x}}\right]\right)$$

$$= \frac{S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}}{\lambda + \hat{\beta}^2}.$$

The variability as explained by $\hat{\sigma}^2$ is proportional to the variability of SS_E. We only consider the case $\lambda = 1$ that is when $\boldsymbol{\Omega}_{11} = \boldsymbol{\Omega}_{22}$. For those cases when $\lambda \neq 1$, we can always reduce it to the case of $\lambda = 1$ by dividing the observed values of y_k by $\sqrt{\lambda_k}$ as the ULFR (Kendall & Stuart, 1979). Hence,

$$SS_E = \frac{S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}}{1 + \hat{\beta}^2} \tag{4.29}$$

Using regression idea, then the coefficient of determination can be defined as

$$R_p^2 = \frac{SS_R}{S_{yy}} = 1 - \frac{SS_E}{S_{yy}} = \frac{S_{yy} - SS_E}{S_{yy}} \tag{4.30}$$

For the case $\lambda = 1$, Equation (4.30) becomes

$$R_p^2 = \frac{\hat{\beta}S_{xy}}{S_{yy}} \tag{4.31}$$

Proof: we need to show $\frac{SS_R}{S_{yy}} = \frac{\hat{\beta}S_{xy}}{S_{yy}} \quad \Leftrightarrow SS_R = \hat{\beta}S_{xy}$.

By definition, $SS_R = S_{yy} - SS_E$

$$= S_{yy} - \left(\frac{S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}}{1 + \hat{\beta}^2}\right)$$

$$= \frac{\left(S_{yy} + \hat{\beta}^2 S_{yy}\right) - \left(S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}\right)}{1 + \hat{\beta}^2}$$

$$= \frac{\hat{\beta}^2 S_{yy} + 2\hat{\beta}S_{xy} - \hat{\beta}^2 S_{xx}}{1 + \hat{\beta}^2}$$

$$= \frac{\hat{\beta}^2\left(S_{yy} - S_{xx}\right) + 2\hat{\beta}S_{xy}}{1 + \hat{\beta}^2} \tag{4.32}$$

From Equation (4.10) and $\lambda = 1$, we have

$$S_{xy}\hat{\beta}^2 = \left(S_{yy} - S_{xx}\right)\hat{\beta} + S_{xy} \tag{4.33}$$

Substitute Equation (4.33) into Equation (4.32) yields

$$SS_R = \frac{\hat{\beta}\left\{\left[\left(S_{yy}-S_{xx}\right)\hat{\beta}+S_{xy}\right]+S_{xy}\right\}}{1+\hat{\beta}^2}$$

$$= \frac{\hat{\beta}\left\{\hat{\beta}^2 S_{xy}+S_{xy}\right\}}{1+\hat{\beta}^2} = \frac{\hat{\beta}S_{xy}\left(\hat{\beta}^2+1\right)}{1+\hat{\beta}^2} = \hat{\beta}S_{xy}$$

Result 6: Let the ratio of the error variances be known and equals one $(\lambda = 1)$, then the coefficient of determination of the MULFR model is

$$R_p^2 = \frac{SS_R}{S_{yy}} = \frac{\hat{\beta}S_{xy}}{S_{yy}}$$

4.4 Properties of Coefficient of Determination when $\lambda = 1$

4.4.1 Range: $0 \le R_p^2 \le 1$ (Boundedness and Nonnegative)

From the regression sum of squares, we have

$$0 \le SS_R = S_{yy} - SS_E \le S_{yy}$$

$$0 \le \frac{SS_R}{S_{yy}} \le \frac{S_{yy}}{S_{yy}} = 1$$

$$\therefore 0 \le R_p^2 \le 1$$

4.4.2 Range of R_p^2 (an improvement of Section 4.4.1)

Let $S_{yy} = kS_{xx}$. Since $S_{yy} \ge 0$ and $S_{xx} \ge 0$, we consider $k > 0$. From Equation (4.31), we have

$$R_p^2 = \frac{\hat{\beta}S_{xy}}{S_{yy}} = \frac{\left(S_{yy}-S_{xx}\right)+\sqrt{\left(S_{yy}-S_{xx}\right)^2+4S_{xy}^2}}{2S_{yy}}$$

$$= \frac{\left(kS_{xx}-S_{xx}\right)+\sqrt{\left(kS_{xx}-S_{xx}\right)^2+4S_{xy}^2}}{2kS_{xx}}$$

$$= \frac{\left(k-1\right)S_{xx}+\sqrt{\left(k-1\right)^2 S_{xx}^2+4S_{xy}^2}}{2kS_{xx}}$$

$$= \frac{(k-1)}{2k} + \sqrt{\frac{(k-1)^2 S_{xx}^2 + 4S_{xy}^2}{4k^2 S_{xx}^2}}$$

$$\geq \frac{(k-1)}{2k} + \sqrt{\frac{(k-1)^2 S_{xx}^2}{4k^2 S_{xx}^2}} \qquad \left(\because 4S_{xy}^2 \geq 0\right)$$

$$= \frac{(k-1)}{2k} + \frac{(k-1)}{2k} = 1 - \frac{1}{k}$$

Since $0 \leq R_p^2 \leq 1$, then $0 \leq 1 - \frac{1}{k} \leq R_p^2 \leq 1$. We consider the following two cases:

Case I: when $0 < k \leq 1$

As $k \to 0^+$, then $R_p^2 \geq 1 - \frac{1}{k} \to -\infty$. However we have $R_p^2 \geq 0$, this implies that $0 \leq R_p^2 \leq 1$.

As $k \to 1^-$, then $R_p^2 \geq 1 - \frac{1}{k} \to 0$. Hence, we have $0 \leq R_p^2 \leq 1$.

As $k = 1$, then $R_p^2 \geq 1 - \frac{1}{k} = 0$. Hence, we have $0 \leq R_p^2 \leq 1$.

Case II: when $k > 1$

As $k \to \infty$, then $R_p^2 \geq 1 - \frac{1}{k} \to 1$. Hence, we have $0 < 1 - \frac{1}{k} \leq R_p^2 \leq 1$.

Result 8: Let $S_{yy} = kS_{xx}$ for $k > 0$ and R_p^2 be the coefficient of determination for MULFR model when $\lambda = 1$. Then

$$\left(1 - \frac{1}{k}\right)^+ \leq R_p^2 \leq 1$$

where $c^+ = \begin{cases} c, & c > 0 \\ 0, & c \leq 0 \end{cases}$ and $c = 1 - \frac{1}{k}$. The full range $0 \leq R_p^2 \leq 1$ is achieved when $0 < k \leq 1$.

4.4.3 Non-Symmetry Property

Given the MULFR model defined by Equations (4.1) and (4.2) with $\lambda = 1$, we have

$$\hat{\beta} = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xy}}$$

and $$R_p^2 = \frac{\hat{\beta} S_{xy}}{S_{yy}} = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{yy}}$$

$$\Rightarrow \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2} = 2S_{yy} R_p^2 - \left(S_{yy} - S_{xx}\right) \tag{4.34}$$

Now we consider a MULFR model by replacing Equation (4.1) with

$$\boldsymbol{X}_i = \boldsymbol{\alpha}^* + \beta^* \boldsymbol{Y}_i, \qquad i = 1, 2, \ldots, n \tag{4.35}$$

It can be shown that the estimated slope $\left(\hat{\beta}^*\right)$ and coefficient of determination, say $\tilde{R}_p^2$ for the new model when $\lambda = 1$ are

$$\hat{\beta}^* = \frac{\left(S_{xx} - S_{yy}\right) + \sqrt{\left(S_{xx} - S_{yy}\right)^2 + 4S_{xy}^2}}{2S_{xy}}$$

and $$\tilde{R}_p^2 = \frac{\hat{\beta}^* S_{xy}}{S_{xx}} = \hat{\beta}^* = \frac{\left(S_{xx} - S_{yy}\right) + \sqrt{\left(S_{xx} - S_{yy}\right)^2 + 4S_{xy}^2}}{2S_{xx}}$$

$$= \frac{-\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xx}}$$

$$= -\frac{1}{2S_{xx}}\left[\left(S_{yy} - S_{xx}\right) - \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}\right]$$

$$= -\frac{1}{2S_{xx}}\left[\left(S_{yy} - S_{xx}\right) - 2S_{yy} R_p^2 + \left(S_{yy} - S_{xx}\right)\right] \qquad \text{from Equation (4.34)}$$

$$= -\frac{1}{2S_{xx}}\left[2\left(S_{yy} - S_{xx}\right) - 2S_{yy} R_p^2\right]$$

$$= \frac{S_{yy}}{S_{xx}} R_p^2 - \frac{\left(S_{yy} - S_{xx}\right)}{S_{xx}}$$

Let $S_{yy} = kS_{xx}$ and $k > 0$, then $\tilde{R}_p^2 = kR_p^2 - k + 1 = k\left(R_p^2 - 1\right) + 1$ (4.36)

Result 7: Given $S_{yy} = kS_{xx}$ where $k > 0$. Let R_p^2 and $\tilde{R}_p^2$ be the coefficient of determination for MULFR model with $\lambda = 1$ as defined by Equation (4.1) and Equation (4.36), respectively. Then

$$\tilde{R}_p^2 = k\left(R_p^2 - 1\right) + 1 .$$

Hence, R_p^2 is symmetric when $k = 1$ and non-symmetric otherwise.

The failure of satisfying symmetric property does not create much problem to the application of R_p^2 since S_{yy} and S_{xx} can always be determined and conversion between R_p^2 and $\tilde{R}_p^2$ can be easily done. For consistency purpose, we consider the image with larger variance as ***X*** and the image with smaller variance as ***Y***. With this arrangement, the full range $0 \le R_P^2 \le 1$ will also be granted.

4.4.4 Identity of Indiscernible (Self-Distance)

Let $\boldsymbol{x} = \boldsymbol{y}$, then $R_p^2 = \dfrac{\left(S_{yy} - S_{yy}\right) + \sqrt{\left(S_{yy} - S_{yy}\right)^2 + 4S_{yy}^2}}{2S_{yy}} = \dfrac{\sqrt{4S_{yy}^2}}{2S_{yy}} = 1$.

Let $R_p^2 = 1$, then $\dfrac{\hat{\beta} S_{xy}}{S_{yy}} = 1$

$$\Rightarrow \hat{\beta} = \frac{S_{yy}}{S_{xy}} = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xy}}$$

$$2S_{yy} = \left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}$$

$$S_{yy} + S_{xx} = \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}$$

$$S_{yy}^2 + S_{xx}^2 + 2S_{xx}S_{yy} = S_{yy}^2 + S_{xx}^2 - 2S_{xx}S_{yy} + 4S_{xy}^2$$

$$4S_{xx}S_{yy} = 4S_{xy}^2 \quad \Rightarrow \boldsymbol{x} = \boldsymbol{y}$$

Therefore, $R_p^2 = 1 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$.

4.4.5 Translation Invariant

Given $S_{xy} = \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i - n\bar{\boldsymbol{x}}'\bar{\boldsymbol{y}}$ and ***a*** and ***b*** be constant vectors.

$$S_{(x+a)(y+b)} = \sum_{i=1}^{n} (\boldsymbol{x}_i + \boldsymbol{a})' (\boldsymbol{y}_i + \boldsymbol{b}) - \frac{1}{n} \sum_{i=1}^{n} (\boldsymbol{x}_i + \boldsymbol{a})' \sum_{i=1}^{n} (\boldsymbol{y}_i + \boldsymbol{b})$$

$$= \sum_{i=1}^{n} \left(\boldsymbol{x}_i' \boldsymbol{y}_i + \boldsymbol{a}' \boldsymbol{y}_i + \boldsymbol{b}' \boldsymbol{x}_i + \boldsymbol{a}' \boldsymbol{b} \right) - \frac{1}{n} \left(\sum_{i=1}^{n} \boldsymbol{x}_i' + n\boldsymbol{a}' \right) \left(\sum_{i=1}^{n} \boldsymbol{y}_i + n\boldsymbol{b} \right)$$

$$= \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i + \boldsymbol{a}' \sum_{i=1}^{n} \boldsymbol{y}_i + \boldsymbol{b}' \sum_{i=1}^{n} \boldsymbol{x}_i + n\boldsymbol{a}'\boldsymbol{b} - \frac{1}{n} \left[\left(\sum_{i=1}^{n} \boldsymbol{x}_i' \right) \left(\sum_{i=1}^{n} \boldsymbol{y}_i \right) + n\boldsymbol{a}' \sum_{i=1}^{n} \boldsymbol{y}_i + n\boldsymbol{b}' \sum_{i=1}^{n} \boldsymbol{x}_i + n^2 \boldsymbol{a}'\boldsymbol{b} \right]$$

$$= \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i + \boldsymbol{a}' \sum_{i=1}^{n} \boldsymbol{y}_i + \boldsymbol{b}' \sum_{i=1}^{n} \boldsymbol{x}_i + n\boldsymbol{a}'\boldsymbol{b} - \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i' \sum_{i=1}^{n} \boldsymbol{y}_i - \boldsymbol{a}' \sum_{i=1}^{n} \boldsymbol{y}_i - \boldsymbol{b}' \sum_{i=1}^{n} \boldsymbol{x}_i - n\boldsymbol{a}'\boldsymbol{b}$$

$$= \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i - \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i' \sum_{i=1}^{n} \boldsymbol{y}_i$$

$$= \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}} = S_{xy}$$

Follow the same argument, we obtain $S_{(x+a)(x+b)} = S_{xx}$ and $S_{(y+a)(y+b)} = S_{yy}$. Similarly, we can show that

$$\hat{\beta}_{(x+a)(y+b)} = \frac{\left(S_{(y+a)(y+b)} - S_{(x+a)(x+b)} \right) + \sqrt{\left(S_{(y+a)(y+b)} - S_{(x+a)(x+b)} \right)^2 + 4S_{(x+a)(y+b)}^2}}{2S_{(x+a)(y+b)}}$$

$$= \frac{\left(S_{yy} - S_{xx} \right) + \sqrt{\left(S_{yy} - S_{xx} \right)^2 + 4S_{xy}^2}}{2S_{xy}} = \hat{\beta}_{xy} .$$

Therefore, $R_{(x+a)(y+b)}^2 = \dfrac{\hat{\beta}_{(x+a)(y+b)} S_{(x+a)(y+b)}}{S_{(y+a)(y+b)}} = \dfrac{\hat{\beta}_{xy} S_{xy}}{S_{yy}} = R_{xy}^2$.

4.4.6 Scale Invariant

Given $S_{xy} = \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i - n\overline{\boldsymbol{x}}'\overline{\boldsymbol{y}}$ and *a* and *c* be constant vectors.

$$S_{ax,cy} = \sum_{i=1}^{n} (a\boldsymbol{x}_i)' (c\boldsymbol{y}_i) - \frac{1}{n} \sum_{i=1}^{n} (a\boldsymbol{x}_i)' \sum_{i=1}^{n} (c\boldsymbol{y}_i)$$

$$= ac \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i - \frac{ac}{n} \sum_{i=1}^{n} \boldsymbol{x}_i' \sum_{i=1}^{n} \boldsymbol{y}_i$$

$$= ac\left(\sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{y}_i - \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i' \sum_{i=1}^{n} \boldsymbol{y}_i\right) = acS_{xy} .$$

Similarly, it can be shown that $S_{ax,ax} = a^2 S_{xx}$ and $S_{cy,cy} = c^2 S_{yy}$. This implies that

$$\hat{\beta}_{ax,cy} = \frac{\left(S_{cy,cy} - S_{ax,ax}\right) + \sqrt{\left(S_{cy,cy} - S_{ax,ax}\right)^2 + 4S_{ax,cy}^2}}{2S_{ax,cy}}$$

$$= \frac{\left(c^2 S_{yy} - a^2 S_{xx}\right) + \sqrt{\left(c^2 S_{yy} - a^2 S_{xx}\right)^2 + 4a^2c^2 S_{xy}^2}}{2acS_{xy}}$$

$$= \frac{\left(\frac{c}{a} S_{yy} - \frac{a}{c} S_{xx}\right) + \sqrt{\left(\frac{c}{a} S_{yy} - \frac{a}{c} S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xy}} .$$

Therefore,

$$R_p^2\left(a\boldsymbol{x}, c\boldsymbol{y}\right) = \frac{\hat{\beta}_{ax,cy} S_{ax,cy}}{S_{cy,cy}} = \frac{acS_{xy}}{c^2 S_{yy}} \left\{ \frac{\left(\frac{c}{a} S_{yy} - \frac{a}{c} S_{xx}\right) + \sqrt{\left(\frac{c}{a} S_{yy} - \frac{a}{c} S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xy}} \right\} .$$

$$= \frac{S_{xy}}{S_{yy}} \left\{ \frac{\left(S_{yy} - \frac{a^2}{c^2} S_{xx}\right) + \sqrt{\left(S_{yy} - \frac{a^2}{c^2} S_{xx}\right)^2 + 4\left(\frac{a}{c}\right)^2 S_{xy}^2}}{2S_{xy}} \right\} .$$

When $a = c$, then

$$R_p^2\left(a\boldsymbol{x}, c\boldsymbol{y}\right) = \frac{S_{xy}}{S_{yy}} \left\{ \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{xy}} \right\} = \frac{\hat{\beta}_{xy} S_{xy}}{S_{yy}} = R_P^2\left(\boldsymbol{x}, \boldsymbol{y}\right) = R_P^2 .$$

When $a > c$ or $a < c$, $R_p^2\left(a\boldsymbol{x}, c\boldsymbol{y}\right) \neq R_P^2$.

4.4.7 Confident Interval

Note that $E\left(R_p^2\right) = E\left(\frac{\hat{\beta} S_{xy}}{S_{yy}}\right) = \frac{S_{xy}}{S_{yy}} E\left(\hat{\beta}\right) \doteq \frac{\beta S_{xy}}{S_{yy}}$

and $$Var\left(R_p^2\right)=Var\left(\frac{\hat{\beta}S_{xy}}{S_{yy}}\right)=\frac{S_{xy}^2}{S_{yy}^2}V\hat{a}r\left(\hat{\beta}\right).$$

Refer to Equation (4.23) and let $L_\beta=\hat{\beta}-Z_{\frac{a}{2}}se\left(\hat{\beta}\right)$ and $U_\beta=\hat{\beta}+Z_{\frac{a}{2}}se\left(\hat{\beta}\right)$. There are four possible cases:

Case I: when $U_\beta \geq L_\beta \geq 0$

Then the $(1-a)100\%$ confidence interval for the population R_p^2 is

$$\hat{\beta}\frac{S_{xy}}{S_{yy}}-Z_{\frac{a}{2}}\sqrt{\frac{S_{xy}^2}{S_{yy}^2}V\hat{a}r\left(\hat{\beta}\right)}\leq \beta\frac{S_{xy}}{S_{yy}}\leq \hat{\beta}\frac{S_{xy}}{S_{yy}}+Z_{\frac{a}{2}}\sqrt{\frac{S_{xy}^2}{S_{yy}^2}V\hat{a}r\left(\hat{\beta}\right)}$$

$$\left[\hat{\beta}-Z_{\frac{a}{2}}se\left(\hat{\beta}\right)\right]\frac{S_{xy}}{S_{yy}}\leq \beta\frac{S_{xy}}{S_{yy}}\leq\left[\hat{\beta}+Z_{\frac{a}{2}}se\left(\hat{\beta}\right)\right]\frac{S_{xy}}{S_{yy}}$$

$$\frac{L_\beta S_{xy}}{S_{yy}}\leq \beta\frac{S_{xy}}{S_{yy}}\leq\frac{U_\beta S_{xy}}{S_{yy}} \tag{4.37}$$

Case II: When $U_\beta > 0 > L_\beta$ and $\left|U_\beta\right| \geq \left|L_\beta\right|$, then Equation (4.37) holds.

Case III: When $U_\beta > 0 > L_\beta$ and $\left|U_\beta\right| < \left|L_\beta\right|$, then $\frac{U_\beta S_{xy}}{S_{yy}}\leq \beta\frac{S_{xy}}{S_{yy}}\leq\frac{L_\beta S_{xy}}{S_{yy}}$ (4.38)

Case IV: When $0 \geq U_\beta \geq L_\beta$, then Equation (4.38) holds.

Result 9: Let the ratio of the error covariances be known and equals one $(\lambda=1)$, then the $(1-a)100\%$ confidence interval for the population R_p^2 is

$$\frac{L_\beta S_{xy}}{S_{yy}}\leq \beta\frac{S_{xy}}{S_{yy}}\leq\frac{U_\beta S_{xy}}{S_{yy}} \qquad \text{if } \left|U_\beta\right| \geq \left|L_\beta\right|$$

$$\frac{U_\beta S_{xy}}{S_{yy}}\leq \beta\frac{S_{xy}}{S_{yy}}\leq\frac{L_\beta S_{xy}}{S_{yy}} \qquad \text{if } \left|U_\beta\right| < \left|L_\beta\right|.$$

4.4.8 R_F^2 is a Special Case of R_p^2 When $p=1$.

From Equation (3.22), the coefficient of determination for ULFR model when $\lambda=1$ is

$$R_F^2 = \frac{\left(S_{yy} - \lambda S_{xx}\right) + \left\{\left(S_{yy} - \lambda S_{xx}\right)^2 + 4\lambda S_{xy}\right\}^{1/2}}{2S_{yy}}$$

where $S_{yy} = \sum y_i^2 - n\bar{y}^2$, $S_{xx} = \sum x_i^2 - n\bar{x}^2$ and $S_{xy} = \sum x_i y_i - n\bar{x}\bar{y}$.

From Result 3 of MULFR model, we have

$$S_{\boldsymbol{xx}} = \sum_{i=1}^{n} \boldsymbol{x}_i' \boldsymbol{x}_i - n\bar{\boldsymbol{x}}'\bar{\boldsymbol{x}}$$

$$= \sum \begin{bmatrix} x_{1i} & x_{2i} & \cdots & x_{pi} \end{bmatrix} \begin{bmatrix} x_{1i} & x_{2i} & \cdots & x_{pi} \end{bmatrix}' - n \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p \end{bmatrix} \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_p \end{bmatrix}'$$

$$= \sum \left(x_{1i}^2 + x_{2i}^2 + \cdots + x_{pi}^2\right) - n\left(\bar{x}_1^2 + \bar{x}_2^2 + \cdots + \bar{x}_p^2\right)$$

$$= \left(\sum x_{1i}^2 - n\bar{x}_1^2\right) + \left(\sum x_{2i}^2 - n\bar{x}_2^2\right) + \cdots + \left(\sum x_{pi}^2 - n\bar{x}_p^2\right)$$

$$= S_{xx}^1 + S_{xx}^2 + \cdots + S_{xx}^p$$

When $p = 1$, then $S_{\boldsymbol{xx}} = S_{xx}^1 = S_{xx}$. Similarly, we have $S_{\boldsymbol{yy}} = S_{yy}^1 = S_{yy}$ and $S_{\boldsymbol{xy}} = S_{xy}^1 = S_{xy}$.

Therefore, $R_p^2 = \dfrac{\left(S_{\boldsymbol{yy}} - S_{\boldsymbol{xx}}\right) + \sqrt{\left(S_{\boldsymbol{yy}} - S_{\boldsymbol{xx}}\right)^2 + 4S_{\boldsymbol{xy}}^2}}{2S_{\boldsymbol{yy}}}$

$$\Rightarrow R_{p=1}^2 = \frac{\left(S_{yy} - S_{xx}\right) + \sqrt{\left(S_{yy} - S_{xx}\right)^2 + 4S_{xy}^2}}{2S_{yy}} = R_F^2$$

Result 10: Given that $\lambda = 1$. Let R_F^2 and R_p^2 be the coefficient of determination for ULFR model and p-dimensional MULFR model, respectively. When $p = 1$, then

$$R_{p=1}^2 = R_F^2$$

4.5 Conclusion

A new correlation-based statistical similarity measure was proposed in this chapter. It is a generalization from the ULFR model and the similarity measure, R_p^2, is non-symmetric under certain condition stated in Result 8. The proposed measure R_p^2 has many potential applications in image processing such as image quality assessment, performance evaluation for image processing algorithms and pattern recognition. It measures the proportion of variation in reference image explained by the distorted image. The advantages of the proposed measure as compare to other existing similarity measures are: (i) it accommodates to both perfect and non-perfect reference image, (ii) it has flexibility in combining one or more image quality features, and (iii) it compromised between global similarity measure and localized similarity measure. In the next chapters, some simulation works will be carried out to verify the properties of the MULFR model with single slope. Then, we apply the proposed R_p^2 as a performance indicator for feature vectors matching in character recognition and image quality assessment of JPEG compression.